DAMTP/96-101
hep-th/9612082

String/Quantum Gravity motivated Uncertainty Relations and Regularisation in Field Theory*

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1 Overview

The uncertainty relations and the underlying canonical commutation relations are at the heart of quantum mechanics. In recent years, for various conceptual and technical reasons, there has been renewed interest in the question whether there may exist corrections to the canonical commutation relations which could be significant at extreme scales. Let me start with a brief (and certainly incomplete) overview.

Probably the most general approach is the ansatz of 'generalised quantum dynamics', developed by Adler et al. This framework not only allows for commutation relations of generic form, but also includes a possible generalisation of the normally underlying complex Hilbert space to a quaternionic or octonic space. Within this approach, the ordinary canonical commutation relations have been shown to arise as a first order approximation from a statistical averaging process, see [1].

Studies which suggest specific correction terms to the commutation relations between the position and the momentum operators have appeared in the context of both string theory and quantum gravity. From the quantum gravity point of view it has long been argued that, when attempting the resolution of extremely small distances, the space-time disturbing gravitational effect of the necessarily high energy of the probing particle must eventually pose an ultimate limit to the possible resolution of distances, the latest at the Planck scale. Indeed, in string theory, a number of studies, e.g. on string scattering, have suggested the existence of a finite minimal uncertainty in positions Δx_0, see e.g. [2, 3, 4]. Intuitively, the use of higher energies for probing small scales eventually no longer allows to further improve the spatial resolution since

*To appear in Proc. XXI Int. Coll. on Group Theor. Methods in Physics, Goslar, July 1996

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this energy would enlarge the probed string. The net effect has been found to be a correction to the $\mathbf{x}, \mathbf{p}$ uncertainty relation of the form

$$\Delta x \Delta p \geq \frac{\hbar}{2}(1 + \beta(\Delta p)^2 + ...) \tag{1}$$

Through $\Delta A \Delta B \geq 1/2|\langle[A, B]\rangle|$ then follows $[\mathbf{x}, \mathbf{p}] = i\hbar(1 + \beta\mathbf{p}^2 + ...)$. As is easily checked, Eq.1 implies a finite minimal uncertainty $\Delta x_0 = \hbar\sqrt{\beta}$. For recent reviews see [5, 6]. The existence of a lower bound Δx_0 to the standard deviation of position measurements would be a true quantum structure, in the sense that it has no classical analog. Among its attractive features is that it does not require the breaking of translation invariance. An additional scale dependence of Δx_0, due to time-of-flight effects, has also been suggested, for a recent reference see e.g.[7].

Specific correction terms to the commutation relations among the position operators have also been suggested, in particular, in [8]. A key idea there is, that for optimal spatial resolution in one direction, the energy of the probing particle should be delocalised in orthogonal directions, in order to reduce the gravitationally disturbing energy density at the location of the measurement. Uncertainty relations of the form $\sum_{i>j} \Delta x_i \Delta x_j \geq l_{pl}^2 + ...$ have therefore been suggested. Noncommuting position operators were probably first investigated by Snyder in 1947 [9], in an approach which has been followed since, mainly by Russian schools, see e.g.[10].

Specific correction terms to the commutation relations among the momentum operators have also been suggested, e.g. in [11], with the underlying idea to account for the noncommutativity of translation operators on curved space. These corrections would only be relevant at large scales, i.e. as an infrared effect.

A related field, in the sense that it also involves generalised commutators, is the programme of exploring the possibilities of generalised internal and external symmetries. This field is being pursued intensively in the literature, in the context of noncommutative geometry in the sense of Connes [12], and in the context of quantum groups, to which the special volume of these proceedings is devoted.

2 Feynman rules on generalised geometries

We consider general canonical commutation relations

$$[\mathbf{x}_i, \mathbf{p}_j] = i\hbar \left(\delta_{ij} + \Theta_{ij}(\mathbf{x}, \mathbf{p})\right) \tag{2}$$

where Θ is a not necessarily symmetric function in the $\mathbf{x}_i$ and $\mathbf{p}_i$. Similarily, we allow $[\mathbf{x}_i, \mathbf{x}_j] \neq 0$ and $[\mathbf{p}_i, \mathbf{p}_j] \neq 0$. We require however the standard involution $\mathbf{x}^\dagger = \mathbf{x},\ \mathbf{p}^\dagger = \mathbf{p}$ in order to guarantee real expectation values.

In this way we cover the case of the corrections to the $\mathbf{x}, \mathbf{p}$ commutation relations which we mentioned above, namely which imply a finite minimal uncertainty Δx_0, as

well as the case $[\mathbf{x},\mathbf{p}] = i\hbar(1+\alpha\mathbf{x}^2+\beta\mathbf{p}^2)$ which implies minimal uncertainties Δx_0 and Δp_0 in both positions and momenta, see [13]-[15].

A general framework for the formulation of quantum field theories on such 'noncommutative geometries' has been given in [16, 17]. Consider the example of euclidean charged scalar ϕ^4-theory:

$$Z[J] := N \int D\phi \; e^{-\int d^4x \; [\phi^*(-\partial_i\partial^i + m^2c^2)\phi + \frac{\lambda}{4!}(\phi\phi)^*\phi\phi - \phi^* J - J^*\phi]} \tag{3}$$

For our purpose it is useful to formulate the action functional without reference to any particular choice of basis in the space of fields that is formally being summed over. The functional analytic structure is analogous to the situation in quantum mechanics, with fields being vectors in a representation of the canonical commutation relations. We therefore formally extend the Dirac notation for states to fields, i.e. $\phi(x) = \langle x|\phi\rangle$. Of course, the simple quantum mechanical interpretation of fields $|\phi\rangle$ and in particular of the position and momentum operators does not simply extend. However, this notation clarifies the functional analytic structure of the action functional:

$$Z[J] = N \int D\phi \; e^{-\frac{l^2}{\hbar^2}\; \langle\phi| \; \mathbf{p}^2 + m^2c^2 \; |\phi\rangle \; - \frac{\lambda l^4}{4!}\langle\phi*\phi|\phi*\phi\rangle \; + \langle\phi|J\rangle + \langle J|\phi\rangle} \tag{4}$$

(We introduced a unit of length l which could be reabsorbed in a trivial field redefinition). Ordinarily, when formulating a field theory in position space, as in Eq.3, the fact that $\mathbf{p}^2$ is represented as $-\hbar^2\nabla$ already implies that $\mathbf{p}$ is represented as $-i\hbar\partial_{x_i}$, so that it must obey the ordinary commutation relations. The advantage of the representation independent formulation in Eq.4 is that the underlying commutation relations are not implicitly specified and can therefore be generalised.

We can then derive the Feynman rules in any arbitrary Hilbert basis $\{|n\rangle\}_n$ in the space F of fields on which the generalised commutation relations Eqs.2 are represented. While this basis can be chosen continuous, discrete, or generally a mixture of both, we here use the convenient notation for discrete n. Fields, operators and the pointwise multiplication $*$ of fields are now expanded as

$$\phi_n = \langle n|\phi\rangle \qquad \text{and} \qquad (\mathbf{p}^2 + m^2c^2)_{nm} = \langle n|\mathbf{p}^2 + m^2c^2|m\rangle \tag{5}$$

$$* = \sum_{n_i} L_{n_1,n_2,n_3} |n_1\rangle \otimes \langle n_2| \otimes \langle n_3| \tag{6}$$

We remark that, ordinarily, $(\phi_1 * \phi_2)(x) = \phi_1(x)\phi_2(x)$, i.e. $*$ takes the form:

$$* = \int d^4x \; |x\rangle \otimes \langle x| \otimes \langle x| \tag{7}$$

In the $\{|n\rangle\}$ basis, Eq.4 reads, summing over repeated indices:

$$Z[J] = N \int_F D\phi \; e^{-\frac{l^2}{\hbar^2}\; \phi^*_{n_1}(\mathbf{p}^2+m^2c^2)_{n_1n_2}\phi_{n_2} - \frac{\lambda l^4}{4!} L^*_{n_1n_2n_3} L_{n_1n_4n_5}\phi^*_{n_2}\phi^*_{n_3}\phi_{n_4}\phi_{n_5} + \phi^*_n J_n + J^*_n\phi_n} \tag{8}$$

Pulling the interaction term in front of the path integral, completing the squares, and carrying out the gaussian integrals yields

$$Z[J] = N' e^{-\frac{\lambda l^4}{4!} L^*_{n_1 n_2 n_3} L_{n_1 n_4 n_5} \frac{\partial}{\partial J_{n_2}} \frac{\partial}{\partial J_{n_3}} \frac{\partial}{\partial J^*_{n_4}} \frac{\partial}{\partial J^*_{n_5}}} \; e^{-\frac{\hbar^2}{l^2} J^*_n (\mathbf{p}^2 + m^2 c^2)^{-1}_{nm} J_m} \tag{9}$$

The Feynman rules therefore read, see [16, 17]:

$$\Delta_{nm} = \left(\frac{\hbar^2 / l^2}{\mathbf{p}^2 + m^2 c^2} \right)_{nm}, \qquad \Gamma_{rstu} = -\frac{\lambda l^4}{4!} L^*_{nrs} L_{ntu} \tag{10}$$

On ordinary geometry, i.e. with the ordinary commutation relations $[\mathbf{x}_i, \mathbf{p}_j] = i\hbar\delta_{ij}$ underlying, the choice e.g. of the position eigenbasis $|n\rangle = |x\rangle$ or the momentum eigenbasis $|n\rangle = |p\rangle$ of course recovers the usual formulations of the Feynman rules.

3 Regularisation

Eqs.10 yield the Feynman rules for generic commutation relations, and for arbitrary choices of basis. Let us now consider the question whether the theory is UV and/or IR finite on geometries (commutation relations) which imply a finite minimal uncertainty $\Delta x_0 > 0$ and/or $\Delta p_0 > 0$. Indeed, the following two statements can be made for arbitrary geometries (generalised commutation relations):

(A) $\Delta p_0 > 0 \quad \Rightarrow \quad$ in QM: $\quad ||\frac{1}{\mathbf{p}^2}|| < \infty \quad \Rightarrow \quad$ in QFT: propagator IR regular

Both steps, first that $\Delta p_0 > 0$ implies that the inverse of the operator $\sum_i \mathbf{p}_i^2$ is a bounded self-adjoint operator, and secondly the implementation into field theory have been shown in [17]. A crucial fact is that $\mathbf{p}^2$ becomes positive *definite* on any dense domain on which the commutation relations are represented, if $\Delta p_0 > 0$.

(B) $\Delta x_0 > 0 \quad \Rightarrow \quad$ in QM: $\quad || \, |x^{ml}\rangle|| < \infty \quad \Rightarrow \quad$ in QFT: vertices UV regular

So far, in all known geometries with $\Delta x_0 > 0$ the states of maximal localisation have been found to be normalisable, see [13, 14, 15]. To see that this is generally true, assume $|\psi_n\rangle$ to be a sequence of physical states (i.e. they are in the domain of $\mathbf{x}$ and $\mathbf{p}$) which approximates the vector of maximal localisation, say around the origin: $\lim_{n\to\infty} \Delta x_{|\psi_n\rangle} = \Delta x_0$. It is a sequence within the Hilbert space with respect to the norm induced by the now positive *definite* $\mathbf{x}^2$, and converges therefore towards a vector $|\psi\rangle$ within that Hilbert space. Since the $\mathbf{x}^2$-induced norm is sharper than the original Hilbert space norm, $|\psi\rangle$ is normalisable. Preliminary results on UV regularisation through $\Delta x_0 > 0$ are in [16]. I am currently working out a more comprehensive study with my collaborator G. Mangano [18]. Here is a sketch of the main points.

In Eq.4 the pointwise multiplication $*$ of fields is crucial for the description of local

interaction and is normally given through Eq.7, yielding $(\phi_1 * \phi_2)(x) = \phi_1(x)\phi_2(x)$. On generalised geometries, in order to describe maximally local interactions, Eq.7 can be read with the $|x\rangle$ denoting the vectors of then maximal localisation, i.e.

$$* = \int d^4x \ |x^{ml}\rangle \otimes \langle x^{ml}| \otimes \langle x^{ml}| \tag{11}$$

where $|x^{ml}\rangle$ denote the fields which are maximally localised with position expectation values x, yielding the structure constants $L_{n_1,n_2,n_3} = \int d^4x \ \langle n_1|x^{ml}\rangle\langle x^{ml}|n_2\rangle\langle x^{ml}|n_3\rangle$. As abstract operators, i.e. without specifying a Hilbert basis in the space of fields the free propagator and the lowest order vertex then read, choosing $l := \Delta x_0$:

$$\Delta = \frac{\hbar^2}{(\Delta x_0)^2(\mathbf{p}^2 + m^2c^2)} \tag{12}$$

$$\Gamma = -\frac{\lambda}{4!}\int \frac{d^4x \ d^4y}{(\Delta x_0)^8} \ \ \langle y^{ml}|x^{ml}\rangle \ \ |y^{ml}\rangle \otimes |y^{ml}\rangle \otimes \langle x^{ml}| \otimes \langle x^{ml}| \tag{13}$$

The crucial observation is that in combining the Feynman rules to form graphs, all factors that can appear are either of the type $\langle x^{ml}|y^{ml}\rangle$ or $\langle x^{ml}|(\mathbf{p}^2 + m^2c^2)^{-1}|y^{ml}\rangle$. Both factors are now well behaved and *bounded* functions of x and y, because $1/(\mathbf{p}^2 + m^2)$ is a bounded self-adjoint operator and, crucially, because the $|x^{ml}\rangle$ are normalised. Therefore, all graphs are ultraviolet regular. Of course, as $\Delta x_0 \to 0$, both factors converge towards the distributions as which they are normally defined.

References

[1] S.L. Adler, L.P. Horwitz, Preprint hep-th/9610153

[2] D.J. Gross, P.F. Mende, Nucl.Phys. B303 p.407 (1987)

[3] D. Amati, M. Cialfaloni, G. Veneziano, Phys. Lett. **B216**, 41 (1989)

[4] M. Maggiore, Phys. Lett. **B319**, 83 (1993)

[5] E. Witten, Phys. Today 49 (4), 24 (1996)

[6] L.J. Garay, Int. J. Mod. Phys. **A10**, 145 (1995)

[7] G. Amelino-Camelia, J. Ellis, N.E. Mavromatos, D.V. Nanopoulos, Preprint hep-th/9605211

[8] S. Doplicher, K. Fredenhagen, J.E. Roberts, Comm.Math.Phys. 172 187 (1995)

[9] H.S. Snyder, Phys. Rev. **71**, 38 (1947)

[10] V.G. Kadyshevskii, D.V. Fursaev, Theor.Math.Phys. **83**, 474 (1990)

[11] A. Kempf, to appear in Proc. 'Minisemester on Quantum Groups & Quantum Spaces', Banach Center, Warsaw, Nov.95, hep-th/9603115.

[12] A. Connes, *Noncommutative geometry,* AP (1994)

[13] A. Kempf, J. Math. Phys. 35 (9), 4483 (1994)

[14] A. Kempf, G. Mangano, R.B. Mann, Phys. Rev. **D52**, 1108-1118 (1995)

[15] A. Kempf, H. Hinrichsen, J. Math. Phys. **37**, 2121 (1996)

[16] A. Kempf, Preprint DAMTP/94-33, hep-th/9405067, and DAMTP/96-23, hep-th/9602085, to appear in J. Math. Phys.

[17] A. Kempf, Phys. Rev. **D54**, 5174 (1996)

[18] A. Kempf, G. Mangano, Preprint DAMTP/96-102